M.Sc.(Physics) 3rd Semester Examination 2013

Statistical Mechanics
(PGP - 303)

Time: 3 hours **Full Marks: 70**

Answer any **five** *questions.*

1. (a) (i) Show that for a smooth dynamical system described by a Hamiltonian, phase space points move like an incompressible fluid (Liouville's theorem).

   (ii) Explain briefly Gibbs' idea of ensemble to describe the equilibrium state of a macroscopic system. What is ergodic hypothesis?

   (b) Consider a macroscopic system consisting of N number of very weakly interacting gas molecules occupying volume V with total energy E in equilibrium. Calculate the number of microstates accessible to the system.

   Use this result to obtain entropy and equation of state of the system.

   (c) Show that for two large systems in thermal contact the number of states for the combined state is Gaussian in energy variable of any of the system. Find the root mean square deviation from the mean value for this system.

   [(3+3)+4+4]

2. (a) Consider a system in contact with a very large system (reservoir) with which energy exchange is possible. Treating the combined system as isolated, obtain an expression for ensemble average of a macroscopic quantity related to the system by integrating over the reservoir degrees of freedom.

   Show that the temperature of reservoir is same as the temperature defined for the system in the microcanonical approach to a good approximation.

   (b) Evaluate the canonical partition function for a system of N identical classical harmonic oscillators kept in contact with a reservoir at temperature T. Use this to obtain entropy, pressure and specific heat of the system.

(c) Starting from grand canonical partition function $Q = \sum_N e^{\beta[\mu N - F(N)]}$, where the symbols have usual meaning, obtain the fluctuation in the number of the particles. When does this fluctuation become pronounced?

[4+5+(3+2)]

3. (a) Consider an ensemble of N systems described by Hamiltonian H. Let $\Psi^{(k)}(\mathbf{r}, t)$ be the normalized wave function of the k-th member of the ensemble. Suppose this is expanded in terms of ortho-normal basis states $\{\phi_n(\mathbf{r})\}$ with mixing coefficients $a_n^{(k)}(t)$.

   i) Write down the form of density matrix operator $\hat{\rho}$.

   ii) Show that $i\hbar\frac{d\rho}{dt} = [H, \rho]$

   iii) Obtain the ensemble average of an observable given by operator O.

   (b) Show that at equilibrium the density matrix is diagonal in the basis of energy eigenstates. Use this to construct density matrix of microcanonical ensemble. State clearly the postulates made for this construction.

   (c) Consider a particle of mass m, otherwise free to be confined in a cube of side L. Obtain the canonical density matrix $\rho(\mathbf{r}, \mathbf{r}')$ in coordinate representation. Comment on its limiting form as $T \to 0$ and $T \to \infty$.

[(1+2+3)+(2+2)+4]

4. (a) Show that for an interacting quantum gas the grand canonical partition function can be expressed as

$$\mathcal{Z}(z, V, T) = \exp\left\{\frac{V}{\lambda^3}\sum_{l=1}^{\infty} b_l z^l\right\}$$

   (b) Obtain the expression for second virial coefficient of an interacting quantum gas in terms of bound state energies and the density of states of the continuum.

[7+7]

5. Consider the Ising model in presence of an external magnetic field H.

   (a) Write down the energy for the spin configuration $\{s_i\}$ in a lattice having N sites. Assume that the interaction between any two spins are isotropic and take only into account the nearest neighbour interaction.

   (b) Obtain energy of the given configuration in terms of the variables N_+ and N_{++}, where the symbols have their usual meaning.

(c) Employ Bragg-William approximation to obtain the Curie temperature T_c for paramagnetic to ferromagnetic phase transition. Hence show that the specific heat exhibits discontinuity at $T = T_c$.

[2+3+(4+5)]

6. (a) Consider a magnetic system described by the order parameter density (magnetization) $m(\mathbf{r})$. Show that $\chi = \beta \int d^3r \Gamma(\mathbf{r})$, where

$$\Gamma(\mathbf{r}) = \langle m(\mathbf{r}) m(0) \rangle - \langle m(\mathbf{r}) \rangle \langle m(0) \rangle.$$

(b) Using the property of scale invariance near the point of criticallity, deduce the following scaling relations:

(i) $\nu = 1/D_t$, (ii) $\alpha = 2 - (d/D_t)$, (iii) $\beta = (d - D_h)/D_t$, (iv) $\gamma = (2D_h - d)/D_t$, (v) $\delta = D_h/(d - D_h)$ and (vi) $\eta = d + 2 - 2D_h$.

Here the symbols carry their usual meanings.

[5+(1+8)]

7. (a) Consider the Einstein's theory of Brownian motion in one dimension

(i) Show that the root mean square displacement

$$x_{\text{rms}} \propto \sqrt{t}$$

Discuss the physical significance of this result.

(ii) Show that the probability of finding the particle between x and $x + dx$

$$p(x)dx = \frac{dx}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right)$$

where $D = l^2/2\tau^*$ and the rest symbols have their usual meaning.

(b) Following the Langevin's theory of Brownian motion show that for $t >> \tau$

$$< r^2 >= (6Bk_BT)t$$

where the symbols have their ususal meaning.

[(3+1+3)+7]